## THE UNIVERSITY OF AKRON <br> Mathematics and Computer Science

Lesson 8: Cartesian Coordinate System \& Functions

## Directory

- Table of Contents
- Begin Lesson 8

$$
\operatorname{Iam} \mathfrak{D}
$$

$$
\begin{gathered}
\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \\
a^{3} a^{4}=a^{7}(a b)^{10}=a^{10} b^{10} \\
-(a b-(3 a b-4))=2 a b-4 \\
(a b)^{3}\left(a^{-1}+b^{-1}\right)=(a b)^{2}(a+b) \\
(a-b)^{3}=a^{3}-3 a^{2} b+3 a b^{2}-b^{3} \\
2 x^{2}-3 x-2=(2 x+1)(x-2) \\
\frac{1}{2} x+13=0 \Longrightarrow x=-26 \\
G=\{(x, y) \mid y=f(x)\} \\
f(x)=m x+b
\end{gathered}
$$

Copyright (c)1995-2000 D.P. Story
Last Revision Date: 4/4/2000

Lesson 8: Cartesian Coordinate System \& Functions Table of Contents
8. Cartesian Coordinate System \& Functions 8.1. The Cartesian Coordinate System

- Referencing/Plotting Points • The Distance Formula • The Midpoint of a Line Segment
8.2. Functions
- The Definition - The Domain of a Function - Points of Intersection of Curves


## 8. Cartesian Coordinate System \& Functions

### 8.1. The Cartesian Coordinate System

There are many schemes for referencing points in the plane. Among all these coordinate systems, the Cartesian Coordinate System is the most popular and useful.

$\ddagger$Begin by drawing two real number lines, perpendicular to each other, and intersecting at their zeros. One number line is Figure 1 drawn horizontally, and the other vertically.

The two number lines, called axes, are labeled by some appropriately chosen symbols; usually the horizontal axis is labeled by the letter $x$ and the vertical axis is labeled by the letter $y$. The horizontal axis is called the $x$-axis or the axis of abscissas and the vertical axis is called the $y$-axis or the axis of ordinates.

$\ldots$The two perpendicular axes subdivide the plane into six subsets in such a way that any given point in the plane is either Figure 2 (1) on the $x$-axis, (2) on the $y$-axis, (3) in the first quadrant,
(4) in the second quadrant, (5) in the third quadrant, or (6) in the fourth quadrant. (See Figure 2.)

## - Referencing/Plotting Points

Points in the plane are referenced by their position relative to the two perpendicular axes. We shall not spend a terribly large amount of time on this topic because you no doubt have plotted many points before. We shall be content with reviewing some of the definitions and terminology.

$\ldots$The Method of Referencing a Point. Let $P$ be a point in the plane. Draw a vertical line passing through the point $P$ and Figure 3 a horizontal line through $P$. The vertical line intersects the $x$-axis at a certain position $a$ and the horizontal line intersects the $y$-axis at a certain position $b$. The Cartesian coordinates of the point $P$ is defined as

$$
P(a, b)
$$

Conversely, given ordered pair of numbers, $(a, b)$, there corresponds one and only one point in the plane. This point is the intersection of
the two lines obtained by drawing a vertical line passing through $a$ in the $x$-axis and a horizontal line passing through the number $b$ on the $y$-axis.

Terminology: Let $(a, b)$ is the Cartesian Coordinates of the point $P$. The number $a$ is called the first coordinate of $P$, or the $x$ coordinate of $P$, or the abscissa of $P$; similarly, $b$ is called the second coordinate, or the $y$-coordinate, or the ordinate of $P$.

- Question. Why do you think we have such terminology as the "axis of abscissas," the "axis of ordinates," the "abscissa of P" and the "ordinate of $P$ "?

Having defined the method of referencing a point in the plane, the four quadrants of the plane can be described more precisely.

Quiz. Answer each of the following about the quadrants.

1. What quadrant consists of all points $P(x, y)$ satisfying $x>0$ and $y<0$ ? Quadrant ...
(a) I
(b) II
(c) III
(d) IV
2. What quadrant consists of all points $P(x, y)$ satisfying $x<0$ and $y>0$ ? Quadrant ...
(a) I
(b) II
(c) III
(d) IV
3. What quadrant consists of all points $P(x, y)$ satisfying $x<0$ and $y<0$ ? Quadrant ...
(a) I
(b) II
(c) III
(d) IV
4. What quadrant consists of all points $P(x, y)$ satisfying $x>0$ and $y>0$ ? Quadrant ...
(a) I
(b) II
(c) III
(d) IV

## EndQuiz.

- The Distance Formula

In the next few paragraphs we take up the problem of computing the distance between two points in the plane.

Section 8: Cartesian Coordinate System \& Functions

The Distance Formula:

ELet $P\left(x_{1}, y_{1}\right)$ and $Q\left(x_{2}, y_{2}\right)$ be two points in the plane, then the distance between $P$ and $Q$, denoted Figure 4 by $d(P, Q)$, is given by

$$
\begin{equation*}
d(P, Q)=\sqrt{\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}} \tag{1}
\end{equation*}
$$

The validity of equation (1) is based on the Pythagorean Theorem. From Figure 4, we have

$$
\begin{aligned}
{[d(P, Q)]^{2} } & =\left|x_{1}-x_{2}\right|^{2}+\left|y_{1}-y_{2}\right|^{2} \\
& =\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}
\end{aligned}
$$

whence comes equation (1).
Here is a typical 'beginners' example that illustrates how to use this formula.

Example 8.1. Calculate the distance between the points $P(-2,4)$ and $Q(3,-1)$.

Exercise 8.1. (Skill Level 0) In each of the follow, find the distance, $d(P, Q)$, between $P$ and $Q$. Passing is $100 \%$.
(a) $P(0,0)$ and $Q(3,-4)$ (b) $P(-3,4)$ and $Q(-1,-1)$
(c) $P(-4,2)$ and $Q(5,2)$ (d) $P(-1,5)$ and $Q(7,9)$

With the distance formula firmly in hand, we can now solve a number of problems related to distances in the plane.

Exercise 8.2. Below, we define three points $P, Q$, and $R$. Determine whether these three points are the vertices of a right triangle. (Hint: A triangle is a right triangle if and only if the square of the length of the longest side is equal to the sum of the squares of the other two sides.)
(a) $P(0,0), Q(1,1)$ and $R(2,0)$
(b) $P(6,-7), Q(11,-3)$ and $R(2,-2)$
(c) $P(1,2), Q(-3,4)$ and $R(4,-2)$

The distance formula can also be used to test whether one point is between two others.

Quiz. Let $P, Q$, and $R$ be three points in the plane. Which of the following is equivalent to the statement that $Q$ lies "between" $P$ and $R$; i.e., $Q$ lies on the straight line segment that connects $P$ and $R$. (Hint: Think about the geometry of each equation-draw a picture if necessary.)
(a) $[d(P, R)]^{2}=[d(P, Q)]^{2}+[d(Q, R)]^{2}$
(b) $d(P, R)=d(P, Q)+d(Q, R)$
(c) $[d(P, Q)]^{2}=[d(P, R)]^{2}+[d(R, Q)]^{2}$
(d) $d(P, Q)=d(P, R)+d(R, Q)$

Exercise 8.3. Using the criterion stated in the above Quiz, determine whether $Q$ is between $P$ and $R$.
(a) $P(1,3), Q(2,5)$ and $R(4,9)$
(b) $P(-1,10), Q(2,-5)$, and $R(5,-12)$

Exercise 8.4. The three points $P(1,-1), Q(5,0)$ and $R(3,1)$ are the vertices of a triangle. Compute the perimeter of the triangle $P Q R$.

Exercise 8.5. A particle moves around in the $x y$-plane. It is known that at any given time $t$, the particle has coordinates $P(1+t, 3-t)$. At what time, $t$, will the particle be 4 units away from the origin, $O(0,0)$ ? (Hint: Set up the equation $d(P, O)=4$, and solve for $t$.)

Let us now look at a couple of special cases. Should you go on to Calculus, occasionally you will need to be able to quickly and efficiently compute the distance between two horizontal points and the distance between two vertical points. In these two cases, it is not necessary to use the distance formula in its full generality. Read on.
$\triangleright$ Distance between two Horizontally Oriented Points. Let's begin with a quick quiz, the answer to which represents criterion for judging whether two points are horizontally oriented.

Quiz. Let $P\left(x_{1}, y_{1}\right)$ and $Q\left(x_{2}, y_{2}\right)$ be points in the plane. The $P$ and $Q$ are horizontally oriented is equivalent to the condition
$\begin{array}{ll}\text { (a) } x_{1}=x_{2} & \text { (b) } y_{1}=y_{2}\end{array}$

Now, let us developed a specialized version the distance formula. Let $P\left(x_{1}, y_{1}\right)$ and $Q\left(x_{2}, y_{2}\right)$ be two horizontally oriented points: Then,

$$
\begin{aligned}
d(P, Q) & =\sqrt{\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}} & & \\
& =\sqrt{\left(x_{1}-x_{2}\right)^{2}} & & \triangleleft \text { since } y_{1}=y_{2} \\
& =\left|x_{1}-x_{2}\right| & & \triangleleft \text { by }(1) \text { of LeSSON } 2
\end{aligned}
$$

Distance Between Horizontally Oriented Points: Let $P\left(x_{1}, y_{1}\right)$ and $Q\left(x_{2}, y_{2}\right)$ be two points horizontally oriented points in the plane, i.e. $y_{1}=y_{2}$, then the distance between $P$ and $Q$, denoted by $d(P, Q)$, is given by

$$
\begin{equation*}
d(P, Q)=\left|x_{1}-x_{2}\right| \tag{2}
\end{equation*}
$$

Comment: Usually, equation (2) is evaluated by taking the abscissa of the right-most point and subtracting the abscissa of the left-most point.

Learn to use this specialized formula by answering the questions in the quiz that follows.

Quiz. Work the solutions out first, then choose the correct response. Passing is $100 \%$.

1. Are the points $(2,6)$ and $(-4,6)$ horizontally oriented?
(a) Yes
(b) No
2. Which of the following is the distance between $P(3,-9)$ and $Q(8,-9)$ ?
(a) -11
(b) -5
(c) 5
(d) 11
3. Let $P(6,-3)$ and $Q(-3,-3)$ be given. The value of $d(P, Q)$ is
(a) -9
(b) -3
(c) 3
(d) 9
4. Let $P(-5,2)$ and $Q(-9,2)$ be given. The value of $d(Q, P)$ is
(a) 4
(b) 5
(c) 9
(d) 14

## EndQuiz.

$\triangleright$ Distance between two Vertically Oriented Points. Let $P\left(x_{1}, y_{1}\right)$ and $Q\left(x_{2}, y_{2}\right)$ be two points in the plane. Then $P$ and $Q$ are vertically oriented if and only if $x_{1}=x_{2}$, that is, if they have the same first coordinate.

The Distance Between Vertically Oriented Points: Let $P\left(x_{1}, y_{1}\right)$ and $Q\left(x_{2}, y_{2}\right)$ be two points vertically oriented points in the plane, i.e. $x_{1}=x_{2}$, then the distance between $P$ and $Q$, denoted by $d(P, Q)$, is given by

$$
\begin{equation*}
d(P, Q)=\left|y_{1}-y_{2}\right| \tag{3}
\end{equation*}
$$

Comments: The distance between two vertically oriented points is the height of the upper point minus the height of the lower point.

Exercise 8.6. Use the distance formula to derive equation (3).
Exercise 8.7. Calculate the distance between each of the following sets of points and observe whether each pair of points horizontally or vertically oriented.
(a) $P(1,2)$ and $Q(1,9)$
(b) $P(-3,3)$ and $Q(-3,-4)$
(c) $P(4,3)$ and $Q(-3,3)$
(d) $P(\pi,-5)$ and $Q(\pi,-2)$

## - The Midpoint of a Line Segment

Let $P\left(x_{1}, y_{1}\right)$ and $Q\left(x_{2}, y_{2}\right)$ be two points in the plane. Draw a line segment connecting the point $P$ and $Q$.

Problem. Calculate the Cartesian coordinates of the midpoint of the line segment $P Q$.

$\ldots$
Let us approach this problem as follows ${ }^{1}$ : The coordinate $\bar{x}$ must be halfway between $x_{1}$ and $x_{2}$; the coordinate $\bar{y}$ must be Figure 5 halfway between $y_{1}$ and $y_{2}$. Once we accept this reasoning, we can deduce

$$
\bar{x}=\frac{x_{1}+x_{2}}{2} \quad \bar{y}=\frac{y_{1}+y_{2}}{2} .
$$

(Note: We have used here the midpoint formula for the real number line.) See Figure 5.

[^0]Section 8: Cartesian Coordinate System \& Functions
Thus the Cartesian coordinates of the midpoint are

$$
M\left(\frac{x_{1}+x_{2}}{2}, \frac{y_{1}+y_{2}}{2}\right)
$$

Let's elevate this formula to the status of a shadow box.

Midpoint Formula:
Let $P\left(x_{1}, y_{1}\right)$ and $Q\left(x_{2}, y_{2}\right)$ be points in the plane. The coordinates of the midpoint $M$ between $P$ and $Q$ are given by

$$
\begin{equation*}
M\left(\frac{x_{1}+x_{2}}{2}, \frac{y_{1}+y_{2}}{2}\right) \tag{4}
\end{equation*}
$$

Exercise 8.8. (Skill Level 0) Calculate the midpoint between each of the following pairs of points.
(a) $P(-1,3)$ and $Q(5,7)$
(b) $P(2,4)$ and $Q(2,-5)$
(c) $P(5,-3)$ and $Q(12,-3)$
(d) $P(-1,-1)$ and $Q(4,2)$

Exercise 8.9. Each pair of points listed below is diametrically opposite to each other on a circle. Find the center, $C(h, k)$, and radius, $r$, of the circle.
(a) $P(1,2)$ and $Q(-3,-1)$ (b) $P(-1,4)$ and $Q(0,0)$

### 8.2. Functions

In the world of Mathematics one of the most common creatures encountered is the function. It is important to understand the idea of a function if you want to gain a thorough understanding of Algebra and Calculus.

Science concerns itself with the discovery of physical or scientific truth. In a portion of these investigations, researchers (or engineers) attempt to discern relationships between physical quantities of interest. There are many ways of interpreting the meaning of the word "relationships," but in these lessons we are most often concerned with functional relationships. Roughly speaking, a functional relationship between two variables is a relationship such that one of the two variables
has the property that knowledge of it (or knowledge of its value) implies a knowledge of the value of the other variable.

For example, the physical quantity of area, $A$, of a circle is related to the radius of that circle, $r$. Indeed, it is internationally known that $A=\pi r^{2}$-an equation, I'm sure, you have had more than one occasion to examine in the past. The simple equation $A=\pi r^{2}$ sets forth the principle of a functional relationship: Given knowledge of the value of one variable (the independent variable), $r$, then we have total knowledge of the value of the other variable (the dependent variable), $A$. This causal (or deterministic) relationship one variable has with another variable is the essence of a functional relationship.

This only difference between the example of the previous paragraph and any other example of a function, either one taken from the applied fields or one that is of a more "purely abstract" nature, is the way in which the functional relationship is defined, and the complexity of that definition. There are many, many ways of defining (or describing) a functional relationship between one variable (or a set of variables) and another variable (or another set of variables). Some of these methods
are rather "natural," which you will encounter as you continue with these lessons; others are "unnatural," but we will not encounter them at this level of play.

Before we continue with this discussion, perhaps it is best to have a formalized definition of a function-in the next section.

## - The Definition

Definition. Let $A$ be a set and $B$ be a set. A function, $f$, from $A$ into $B$ is a rule that associates with each element in the set $A$ a unique corresponding element in the set $B$. In this case, we write symbolically, $f: A \rightarrow B$, or $A \xrightarrow{f} B$.

Definition Notes: The set $A$ is called the domain of the function $f$. Typically in Algebra and Calculus, the set $A$ will be an interval of the real number line $\mathbb{R}$. As a notation, we shall refer to the domain of the function $f$ by $\operatorname{Dom}(f)$.

- The set $B$ is called the codomain of $f$. The set $B$ may not be the range of the function.
- Let the elements of $A$ are referred to by the letter $x$, and those of the set $B$ by $y$. The symbol $x$ is called the independent variable of $f$, and $y$ is called the dependent variable. The independent variable can take on any value in the domain, $\operatorname{Dom}(f)$, of $f$.
- A function $f$ is a rule that associates with each element, $x$, in the set $A$, a unique corresponding element, $y$, in the set $B$. The usual way we define a function is by an equation that states the relationship between the variable $x$ and the variable $y$.
- For example, let the function $f$ associate the number $x$ with the number $y$, where $y=x^{2}$. We write,

$$
f: x \rightarrow y \quad \text { where } y=x^{2}
$$

or, more simply

$$
f: x \rightarrow x^{2}
$$

thus,

$$
f: 2 \rightarrow 4 \quad f:-3 \rightarrow 9 \quad f: 0 \rightarrow 0
$$

Where, $f: 2 \rightarrow 4$ states that $f$ associates with $x=2$ the unique corresponding number $y=4$.

- The above notation is used frequently at higher levels of mathematics, at our level, we use the standard functional notation. Rather than writing $f: x \rightarrow x^{2}$, we simply write $f(x)=x^{2}$.

$$
f(x)=x^{2} \text { means } f: x \rightarrow x^{2}
$$

Particular evaluations are carried out as follows:

$$
f(2)=2^{2}=4 \quad f(-3)=(-3)^{2}=9 \quad f(0)=0^{2}=0
$$

More examples are given below.

- Given a particular $x$ in $\operatorname{Dom}(f), y=f(x)$ is called a value of the function $f$. For the function $f(x)=x^{2}$, since $f(2)=4$, we can say that $4(y=4)$ is a value of $f$. It should be clear to you that -4 in not a value of the function $f(x)=x^{2}$.
- For any given function, $f$, some numbers are values of $f$ while others are not. The set of all values of a given function is called its range, denoted by $\operatorname{Rng}(f)$; thus,

$$
\operatorname{Rng}(f)=\{y \mid y \text { is a value of } f\}
$$

To be a value, a ' $y$ ' must be of the form $f(x)$ for some $x$. Thus,

$$
\operatorname{Rng}(f)=\{y \mid y=f(x) \text { for some } x \text { in } \operatorname{Dom}(f)\}
$$

- For the function $f(x)=x^{2}$, the range is

$$
\operatorname{Rng}(f)=[0,+\infty)
$$

Do you understand why?
All the functions that we encounter in Algebra (and most of those encountered in Calculus) defined using algebraic expression in one unknown. For example, the expression $x^{2}-4 x+1$ is an algebraic expression in $x$. We can use this expression to define a function by

$$
f(x)=x^{2}-4 x+1
$$

Here are a few examples of functions defined this way. This method is by no means the only way of defining functions. Read these examples completely and carefully.

Illustration 1. Examples of functions and numerical evaluations.

Section 8: Cartesian Coordinate System \& Functions
(a) Define $f$ by $f(x)=x^{2}+x$. Then,

$$
f(3)=3^{2}+3=12 \quad f(-3)=(-3)^{2}+(-3)=6
$$

(b) Define $g$ by $g(x)=\frac{x}{x+1}$. Then,

$$
g(2)=\frac{2}{2+1}=\frac{2}{3} \quad g\left(-\frac{1}{2}\right)=\frac{\frac{1}{2}}{\frac{1}{2}+1}=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{3} .
$$

The names of functions are determined by the user, that's you and me. I choose a name of $g$ this time.
(c) Define $h$ by $h(t)=\sqrt{t}$. Then,

$$
h(4)=\sqrt{4}=2 \quad h(9)=3 \quad h(5)=\sqrt{5} .
$$

Now I have changed the letter used to denote the independent variable. I have used $t$ instead of the traditional $x$-this causes no problems I hope? Any letter (or symbol) can be used for the independent variable, and any letter (or symbol) can be used for the dependent variable.

Section 8: Cartesian Coordinate System \& Functions
(d) Any symbol you say? How about defining a function $W$ by $W(\phi)=\phi^{3}$. Then,

$$
W(-5)=-125 \quad W(3)=27 \quad W(\sqrt{2})=(\sqrt{2})^{3}=2 \sqrt{2} .
$$

Here, I have used the Greek letter $\phi$ ("phi") for the name of the independent variable.

Evaluation Tip. To evaluate a function, such as $f(x)=x^{2}-2 x$, at a particular value of $x=-1$, first replace the independent variable $x$ with the number -1 , then evaluate the expression. Thus, $f(-1)=$ $(-1)^{2}-2(-1)$ (replace $x$ by -1 ), then evaluate $f(-1)=1+2=3$. Note the use of the parentheses: this is necessary because we are replacing a single letter $x$ by a compound symbol -1 . Not to include these parentheses (1) is mathematically and notationally wrong, and (2) invites evaluation errors.

Exercise 8.10. Evaluate each of the functions defined below at the indicated values. Passing is $100 \%$.
(a) $f(x)=2 x^{2}-3 x ; f(2), f(-2), f\left(-\frac{1}{2}\right)$
(b) $g(s)=s(s+1)(s+2) ; g(0), g(1), g(-1), g(-3)$

Section 8: Cartesian Coordinate System \& Functions

$$
\begin{aligned}
& \text { (c) } h(t)=\frac{t}{t^{2}+1} ; h(1), h(-2), h\left(\frac{1}{2}\right), h\left(-\frac{1}{2}\right) \\
& \text { (d) } D(w)=w \sqrt{w} ; D(9) ; D\left(\frac{1}{9}\right)
\end{aligned}
$$

- For those Who want Greater Insight. Models for Functions. Listed behind this link is a description of several ways in which we can view a function. These points of view may help you to understand this important mathematical object.


## - The Domain of a Function

In the examples in the previous paragraphs, nothing was mentioned concerning the domains of the functions considered. In this section we briefly discuss methods of computing the domain of a function.

The domain of the function defined is either (1) explicitly specified or is (2) not explicitly specified. (That seems reasonable.)

Illustration 2. Examples of functions with explicitly specified domains.

Section 8: Cartesian Coordinate System \& Functions
(a) Define a function $f$ by

$$
f(x)=x^{2}, \quad x \geq 1
$$

Here, we are explicitly defining the domain of $f$ to be

$$
\operatorname{Dom}(f)=[1,+\infty)=\{x \mid x \geq 1\}
$$

For this function, $f(2)=4$ is defined, but $f(0)$ is not because $x=0$ does not fall into the specified domain.
(b) Define a function $g$ by

$$
g(x)=\frac{x}{x^{2}+1}, \quad 0 \leq x<1
$$

Here, we have specified the domain of $g$ to be

$$
\operatorname{Dom}(g)=[0,1)=\{x \mid 0 \leq x<1\}
$$

Illustration Notes: Such (artificial) restriction of the domains may arise from physical considerations. Perhaps these functions, $f$ and $g$ above, are modeling some physical system; within the context of this
physical system, it only make sense to consider $x \geq 1$, in the case of the function $f$, and $0 \leq x<1$ in the case of $g$.
When the domain of a function at definition time is left unspecified, that usually means we are to take as the domain the so-called natural domain of the function.
$\triangleright$ The Natural Domain of a function. Given a function $y=f(x)$. The natural domain of $f$ is the set of all real numbers, $x$, for which the value $f(x)$ can be calculated as a real number.

The next example illustrates the reasoning and methods used to calculate the natural domain of a function. Read carefully!

Example 8.2. Compute the natural domain of each of the following.
(a) $f(x)=x^{2}+3 x+1$
(b) $g(x)=\frac{x^{2}}{x^{2}-3 x+2}$
(c) $h(x)=\sqrt{x+2}$
(d) $p(x)=\sqrt{x^{2}-1}$

Strategy. Let $y=f(x)$ be a function, where $f(x)$ is some algebraic expression. The natural domain consists of all $x$ for which ...

- the denominator (if any) is not equal to zero; and
- any radicands of even roots (if any) are nonnegative.

Here's another example that incorporates all components of the above strategy.
EXAMPLE 8.3. Find the natural domain of $f(x)=\sqrt{\frac{x}{x+1}}$.
Quite typically, the strategy involves setting up some constraints or conditions on the values of the independent variable in the form of inequalities. Once you identify these inequalities, you solve them (possibly using the Sign Chart Method). The natural domain is then the set of all values of the independent variable that satisfy all the constraints or conditions.

Exercise 8.11. Compute the natural domain of each of the following.

$$
\begin{array}{ll}
\text { (a) } f(x)=|x-1| & \text { (b) } g(x)=\frac{3 x}{x^{2}+2 x-8}
\end{array}
$$

Section 8: Cartesian Coordinate System \& Functions

$$
\text { (c) } h(x)=\frac{1}{\sqrt{x(x+2)}} \text { (d) } p(x)=\sqrt{\frac{x}{x^{2}-1}}
$$

## - Points of Intersection of Curves

We have seen in the previous section that determining the natural domain of a functions oftimes require the setting up and solving of inequalities. To determine where two curves intersect, if at all, we must be able to set up and solve equations. This is why the basic mechanics of solving inequalites and equations are so important - equations and inequalities are the natural way in which we ask questions and the techniques of solutions are the way we are able to answer these questions.
$\triangleright$ Determining the $x$-intercept. Let $y=f(x)$ be a function. The $x$ intercept, if there is one, is that value of $x$ such that $f(x)=0$. As you know, every function has a graph-graphing will be taken up in Lesson 9-and in terms of the graph the $x$-intercept is the location on the $x$-axis where the graph crosses the $x$-axis.

Procedure. To find the $x$-intercept(s) of the function $y=f(x)$, set up the equation

$$
\begin{equation*}
f(x)=0 \tag{5}
\end{equation*}
$$

and solve for $x$.
Exercise 8.12. Find the $x$-intercept(s), if any, of each of the following functions.
(a) $f(x)=4 x-1$
(b) $f(x)=x^{2}-3 x+2$
(c) $f(x)=x^{3}+6 x^{2}++8 x$
(d) $f(x)=x^{2}+x+1$
(e) $f(x)=2 x^{2}-x-1$
(f) $f(x)=x^{2}+x-3$

Exercise 8.13. What does the problem of finding the $x$-intercept of a function have to do with the title, "Points of Intersection of Curves," of this section?
$\triangleright$ Determining the intersection of two Curves. Consider the two functions $y=f(x)$ and $y=g(x)$. We wish to find all points, if any, on the intersection of the graphs of $f$ and $g$.

Let $\left(x_{0}, y_{0}\right)$ be a point that is on both graphs of $f$ and $g$. This means

$$
f\left(x_{0}\right)=y_{0} \text { and } g\left(x_{0}\right)=y_{0} .
$$

At this point, we have $f\left(x_{0}\right)=y_{0}=g\left(x_{0}\right)$. This represents a criterion for finding the points of interection of two graphs.

Procedure. Let $y=f(x)$ and $y=g(x)$ be two function. Set up the equation

$$
\begin{equation*}
f(x)=g(x) \tag{6}
\end{equation*}
$$

and solve for $x$.
Finding the points of intersection is essentially a problem in solving equations.

Example 8.4. Find the points of interection beween
(a) $f(x)=3 x+2$ and $g(x)=5 x-4$
(b) $f(x)=x^{2}-3 x+1$ and $g(x)=2 x-5$

Notice that the points of intersection can be calculated without reference to the graph of the functions. At our level of play, finding the points of intersection is purely an exercise in algebra.

Exercise 8.14. Find the cartesian coordinates of the points of intersections of each of the following pairs of functions.
(a) $f(x)=6 x+3$ and $g(x)=2 x-7$
(b) $f(x)=x+3$ and $g(x)=2-8 x$
(c) $f(x)=x^{2}+7 x-1$ and $g(x)=4 x-3$

Now for some exercises that require the use of the Quadratic Formula.

Exercise 8.15. Find the abscissas of intersection of each of the following pairs of functions.
(a) $f(x)=2 x^{2}-5 x+2$ and $g(x)=x+3$
(b) $f(x)=x^{2}+4 x-1$ and $g(x)=1-4 x-x^{2}$
(c) $f(x)=3 x^{2}+1$ and $g(x)=x^{2}-5 x$

Some curves do not intersect. Investigate these kind.
Exercise 8.16. Verify algebraically that each pair of functions do not intersect.
(a) $f(x)=4 x-2$ and $g(x)=4 x+12$
(b) $f(x)=2 x^{2}+3$ and $g(x)=x^{2}-1$
(c) $f(x)=x^{2}+2 x+2$ and $g(x)=x+1$
(d) $f(x)=x^{2}-2 x-4$ and $g(x)=4 x^{2}-3$

We have come to the end of Lesson 8. Congratulations of reaching this far. In Lesson 9, we take up the topics of linear and quadratic functions as well as some graphing topics.

## Solutions to Exercises

8.1. Solutions: I'll just use equation (1).
(a) $P(0,0)$ and $Q(3,-4)$

$$
d(P, Q)=\sqrt{(0-3)^{2}+(0-(-4))^{2}}=\sqrt{9+16}=\sqrt{25}=5
$$

(b) $P(-3,4)$ and $Q(-1,-1)$

$$
\begin{aligned}
d(P, Q) & =\sqrt{(-3-(-1))^{2}+(4-(-1))^{2}} \\
& =\sqrt{(-2)^{2}+5^{2}}=\sqrt{4+25} \\
& =\sqrt{29}
\end{aligned}
$$

(c) $P(-4,2)$ and $Q(5,2)$

$$
\begin{aligned}
d(P, Q) & =\sqrt{(-4-5)^{2}+(2-2)^{2}} \\
& =\sqrt{(-9)^{2}}=|-9| \quad \triangleleft \text { Recall, } \sqrt{x^{2}}=|x| . \\
& =9
\end{aligned}
$$

Solutions to Exercises (continued)
(d) $P(-1,5)$ and $Q(7,9)$

$$
\begin{aligned}
d(P, Q) & =\sqrt{(-1-7)^{2}+(5-9)^{2}} \\
& =\sqrt{(-8)^{2}+(-4)^{2}}=\sqrt{64+16} \\
& =\sqrt{80}=4 \sqrt{5}
\end{aligned}
$$

Exercise 8.1.
8.2. Solutions:
(a) Given $P(0,0), Q(1,1)$ and $R(2,0)$, does $P Q R$ form a right triangle?
Solution: Verify the following calculation:

$$
d(P, Q)=\sqrt{2} \quad d(P, R)=2 \quad d(Q, R)=\sqrt{2}
$$

Note that $\overline{P R}$ is the longest side and

$$
d(P, R)^{2}=d(P, Q)^{2}+d(Q, R)^{2}
$$

Thus, the points $P, Q$, and $R$ form a right triangle.
(b) Given $P(6,-7), Q(11,-3)$ and $R(2,-2)$, does $P Q R$ form a right triangle?
Solution: Verify the following calculations:

$$
d(P, Q)=\sqrt{41} \quad d(P, R)=\sqrt{41} \quad d(Q, R)=\sqrt{82}
$$

The side $\overline{Q R}$ is the longest side. Note that

$$
d(Q, R)^{2}=d(P, Q)^{2}+d(P, R)^{2}
$$

Thus, the points $P, Q$, and $R$ form a right triangle.
(c) $P(1,2), Q(-3,4)$ and $R(4,-2)$

Solution: Verify the following calculations:

$$
d(P, Q)=\sqrt{20}=2 \sqrt{5} \quad d(P, R)=5 \quad d(Q, R)=\sqrt{85}
$$

The side $\overline{Q R}$ is the longest side. Note that

$$
d(Q, R)^{2}=85 \neq 45=20+25=d(P, Q)^{2}+d(P, R)^{2}
$$

The square of the longest side does not equal to the sum of the squares of the other two sides; therefore, $P, Q$ and $R$ do not form a right triangle.
8.3. Solutions:
(a) Given $P(1,3), Q(2,5)$ and $R(4,9)$, is $Q$ between $P$ and $R$ ? Solution:

$$
d(P, Q)=\sqrt{5} \quad d(P, R)=\sqrt{45} \quad d(Q, R)=\sqrt{20}
$$

(Verify these calculations.) Is it true that

$$
\begin{aligned}
d(P, R) & \stackrel{?}{=} d(P, Q)+d(Q, R) \\
\sqrt{45} & \stackrel{?}{=} \sqrt{5}+\sqrt{20} \quad \triangleleft \text { Not obvious! Simplify! } \\
3 \sqrt{5} & \stackrel{?}{=} \sqrt{5}+2 \sqrt{5}
\end{aligned}
$$

They are equal! Indeed, $Q$ does lie between $P$ and $R$.
(b) Given $P(-1,10), Q(2,-5)$, and $R(5,-12)$, is $Q$ between $P$ and $R$ ?
Solution:

$$
d(P, Q)=\sqrt{234} \quad d(P, R)=\sqrt{520} \quad d(Q, R)=\sqrt{58}
$$

(Verify these calculations.) Is it true that

$$
\begin{aligned}
d(P, R) & \stackrel{?}{=} d(P, Q)+d(Q, R) \\
\sqrt{520} & \stackrel{?}{=} \sqrt{234}+\sqrt{58}
\end{aligned}
$$

Make a calculator calculation to see that

$$
\sqrt{520} \neq \sqrt{234}+\sqrt{58}
$$

In this case, the left hand side is not equal to the right hand side. Conclusion: $Q$ is not between $P$ and $R$.

Solutions to Exercises (continued)
8.4. Solution: The perimeter of a triangle is the sum of the lengths of its sides.

Verify the following calculations, and e-mail if I am in error.

$$
d(P, Q)=\sqrt{17} \quad d(Q, R)=\sqrt{5} \quad d(R, P)=2 \sqrt{2}
$$

The perimeter is

$$
\text { perimeter }=\sqrt{17}+\sqrt{5}+2 \sqrt{2}
$$

Question. Is this triangle a right-triangle?
(a) Yes
(b) No

Solutions to Exercises (continued)
8.5. Solution: I'll take my own hint-I hope you did too.

Given: $P(1+t, 3-t)$ and $O(0,0)$

$$
d(P, O)=\sqrt{(1+t)^{2}+(3-t)^{2}}
$$

We want $d(P, O)=4$, therefore,

$$
\text { Solve for } \mathrm{t}: \sqrt{(1+t)^{2}+(3-t)^{2}}=4
$$

Square both sides of the equation, expand and combine.

$$
\begin{aligned}
(1+t)^{2}+(3-t)^{2} & =16 & & \triangleleft \text { square both sides } \\
\left(1+2 t+t^{2}\right)+\left(9-6 t+t^{2}\right) & =16 & & \triangleleft \text { expand } \\
2 t^{2}-4 t+10 & =16 & & \triangleleft \text { combine } \\
t^{2}-2 t+5 & =8 & & \triangleleft \text { divide both sides by } 2 \\
t^{2}-2 t-3 & =0 & & \triangleleft \text { subtract } 18 \text { from both sides }
\end{aligned}
$$

We now solve the equation $t^{2}-2 t-3=0$. We could use the QuAdratic Formula; however, this is an equation that can be solved by factoring:

$$
\begin{aligned}
t^{2}-2 t-3 & =0 \\
(t-3)(t+1) & =0
\end{aligned}
$$

Therefore,

$$
t=-1,3
$$

Presentation of Solution: $t=-1,3$
There are actually two times at which the particle is exactly 4 units away from the origin.

Solutions to Exercises (continued)
8.6. Demonstration: Since the points are vertically oriented, $x_{1}=x_{2}$ :

$$
\begin{aligned}
d(P, Q) & =\sqrt{\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}} & & \\
& =\sqrt{\left(y_{1}-y_{2}\right)^{2}} & & \triangleleft \text { since } x_{1}=x_{2} \\
& =\left|y_{1}-y_{2}\right| & & \triangleleft \text { by }(1) \text { of LeSSON } 2
\end{aligned}
$$

Exercise 8.6.
8.7. Solution: We use equation (3) throughout these solutions.
(a) Given: $P(1,2)$ and $Q(1,9)$, compute $d(P, Q)$. The first coordinates are equal; therefore, these are vertically oriented points. The distance between them is the upper most minus the lower most:

$$
d(P, Q)=|2-9|=9-2=7
$$

(b) Given: $P(-3,3)$ and $Q(-3,-4)$, compute $d(P, Q)$. These points are vertically oriented because the $x$-coordinates are equal.

$$
d(P, Q)=|3-(-4)|=7
$$

(c) Given: $P(4,3)$ and $Q(-3,3)$, compute $d(P, Q)$. Here, the second coordinates are equal; these are horizontally oriented points.

$$
d(P, Q)=|4-(-3)|=7
$$

(d) Given: $P(\pi,-5)$ and $Q(\pi,-2)$, compute $d(P, Q)$. Vertically oriented points.

$$
d(P, Q)=|-5-(-2)|=3
$$

Solutions to Exercises (continued)
What do you know about that, they're all 7 units apart! Oops! All but one - how did that one get in there!

Exercise 8.7.■
8.8. Solutions: Hopefully you used formula (4).
(a) Find the midpoint between $P(-1,3)$ and $Q(5,7)$.

$$
M\left(\frac{-1+5}{2}, \frac{3+7}{2}\right)=M(2,5)
$$

(b) Find the midpoint between $P(2,4)$ and $Q(2,-5)$.

$$
M\left(\frac{2+2}{2}, \frac{4+(-5)}{2}\right)=M\left(2,-\frac{1}{2}\right)
$$

Did you note that these two points were vertically oriented? You did, didn't you.
(c) Find the midpoint between $P(5,-3)$ and $Q(12,-3)$.

$$
M\left(\frac{5+12}{2}, \frac{-3+(-3)}{2}\right)=M\left(\frac{17}{2},-3\right)
$$

These were two horizontally oriented points.

Solutions to Exercises (continued)
(d) Find the midpoint between $P(-1,-1)$ and $Q(4,2)$.

$$
M\left(\frac{-1+4}{2}, \frac{-1+2}{2}\right)=M\left(\frac{3}{2}, \frac{1}{2}\right)
$$

Exercise 8.8. -
8.9. Solutions:
(a) Find the center and radius of the circle containing the points $P(1,2)$ and $Q(-3,-1)$, which are diametrically opposite. Calculation of the Center: The radius is the midpoint of the line segment $P Q$, since the points lie of the same diameter.

$$
C\left(\frac{1+(-3)}{2}, \frac{2+(-1)}{2}\right)=C\left(-1, \frac{1}{2}\right) \quad \triangleleft \text { midpoint formula }
$$

Radius Calculation: The radius is one-half the diameter. Thus, from the distance formula, we have

$$
\begin{aligned}
r & =\frac{1}{2} d(P, Q)=\frac{1}{2} \sqrt{(1-(-3))^{2}+(2-(-1))^{2}} \\
& =\frac{1}{2} \sqrt{4^{2}+3^{2}}=\frac{1}{2} \sqrt{25}=\frac{1}{2} \cdot 5=\frac{5}{2} .
\end{aligned}
$$

Presentation of Answer:

$$
\text { Center: } C\left(-1, \frac{1}{2}\right) \quad \text { Radius: } r=\frac{5}{2}
$$

(b) Find the center and radius of the circle containing the points $P(-1,4)$ and $Q(0,0)$, which are diametrically opposite.
Calculation of the Center: Again, the center is midway between the endpoints of any of its diameters.

$$
C\left(\frac{-1}{2}, \frac{4}{2}\right)=C\left(-\frac{1}{2}, 2\right) \quad \triangleleft \text { midpoint formula }
$$

Calculation of the radius: The radius is one-half the diameter. By the distance formula, we have

$$
r=\frac{1}{2} d(P, Q)=\frac{1}{2} \sqrt{(-1)^{2}+4^{2}}=\frac{1}{2} \sqrt{17 .}
$$

Presentation of Answers:

$$
\text { Center: } C\left(-\frac{1}{2}, 2\right) \quad \text { Radius: } r=\frac{1}{2} \sqrt{17}
$$

8.10. Solutions: Just use the replacement technique. Be sure to verify these calculations.
(a) Given $f(x)=2 x^{2}-3 x$; evaluate $f(2), f(-2), f\left(-\frac{1}{2}\right)$.

$$
\begin{aligned}
f(2) & =2 \cdot 2^{2}-3 \cdot 2=2 \\
f(-2) & =2(-2)^{2}-3(-2)=14 \\
f\left(-\frac{1}{2}\right) & =2\left(-\frac{1}{2}\right)^{2}-3\left(-\frac{1}{2}\right)=2
\end{aligned}
$$

(b) Given $g(s)=s(s+1)(s+2)$; evaluate $g(0), g(1), g(-1), g(-3)$.

$$
\begin{aligned}
g(0) & =0 \\
g(1) & =(1+1)(1+2)=6 \\
g(-1) & =0 \\
g(-3) & =(-3)(-3+1)(-3+2) \\
& =-3(-2)(-1)=-6
\end{aligned}
$$

(c) Given $h(t)=\frac{t}{t^{2}+1}$; evaluate $h(1), h(-2), h\left(\frac{1}{2}\right), h\left(-\frac{1}{2}\right)$.

$$
\begin{aligned}
h(1) & =\frac{1}{1+1}=\frac{1}{2} \\
h(-2) & =\frac{-2}{(-2)^{2}+1}=\frac{-2}{4+1}=-\frac{2}{5} \\
h\left(\frac{1}{2}\right) & =\frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^{2}+1}=\frac{\frac{1}{2}}{\frac{1}{4}+1} \\
& =\frac{\frac{1}{2}}{\frac{5}{4}}=\frac{4}{2 \cdot 5}=\frac{4}{10}=\frac{2}{5} \\
h\left(-\frac{1}{2}\right) & =\frac{-\frac{1}{2}}{\left(-\frac{1}{2}\right)^{2}+1}=-\frac{\frac{1}{2}}{\frac{1}{4}+1}=-\frac{2}{5}
\end{aligned}
$$

(d) Given $D(w)=w \sqrt{w}$; evaluate $D(9) ; D\left(\frac{1}{9}\right)$.

$$
D(9)=9 \sqrt{9}=9 \cdot 3=27 \quad D\left(\frac{1}{9}\right)=\frac{1}{9} \sqrt{\frac{1}{9}}=\frac{1}{9} \cdot \frac{1}{3}=\frac{1}{27}
$$

Solutions to Exercises (continued)
8.11. Solution to: (a) Find the domain of $f(x)=|x-1|$. Here, the natural domain is all of $\mathbb{R}$, the real number line.

$$
\operatorname{Dom}(f)=\mathbb{R}
$$

This is because there are no constraints on the value of $x$. For any $x$ we can calculate $x-1$ and then calculate its absolute value $|x-1|$.

Solutions to Exercises (continued)
Solution to (b) Find the domain of $g(x)=\frac{3 x}{x^{2}+2 x-8}$. In this problem, we just must make sure the denominator is never equal to zero.

$$
\text { Constraints on domain: } x^{2}+2 x-8 \neq 0
$$

To identify these $x$ 's, we find where $x^{2}+2 x-8=0$.
Solve: $\quad x^{2}+2 x-8=0 \quad \triangleleft$ given

$$
\begin{aligned}
(x+4)(x-2) & =0 & & \triangleleft \text { factor } \\
x & =-4,2 & & \triangleleft \text { the solutions }
\end{aligned}
$$

Presentation of Solution:

$$
\begin{aligned}
\operatorname{Dom}(g) & =\left\{x \mid x^{2}+2 x-8 \neq 0\right\} & & \triangleleft \text { initial description } \\
& =\{x \mid x \neq-4 \text { and } x \neq 2\} & & \triangleleft \text { set notation } \\
& =(-\infty,-4) \cup(-4,2) \cup(2,+\infty) & & \triangleleft \text { interval notation }
\end{aligned}
$$

Solutions to Exercises (continued)
Solution to (c) Find the domain of $h(x)=\frac{1}{\sqrt{x(x+2)}}$. Based on the strategy, we see there are a number of constraints on the values of $x$ :

$$
x \neq 0 \quad x \neq-2 \quad x(x+2) \geq 0
$$

These three can be summarized by a single inequality:

$$
\text { Constraints on domain: } x(x+2)>0
$$

We use the Sign Chart Method to analyze $x(x+2)$.
The Sign Chart of $x(x+2)$
$\xrightarrow[-2]{\perp} x+2$


Solutions to Exercises (continued)

Presentation of Solution:

$$
\begin{aligned}
\operatorname{Dom}(h) & =\{x \mid x(x+2)>0\} & & \triangleleft \text { initial description } \\
& =\{x \mid x<-2 \text { or } x>0\} & & \triangleleft \text { set notation } \\
& =(-\infty,-2) \cup(0,+\infty) & & \triangleleft \text { interval notation }
\end{aligned}
$$

Solutions to Exercises (continued)
Solution to (d) Find the domain of $p(x)=\sqrt{\frac{x}{x^{2}-1}}$. From the basic strategy, we see that

$$
\frac{x}{x^{2}-1} \geq 0 \quad x^{2}-1 \neq 0
$$

Realizing that $x^{2}-1 \neq 0$ is equivalent to $x \neq-1$ and $x \neq 1$ is see $\ldots$

$$
\text { Constraints on domain: } \frac{x}{x^{2}-1} \geq 0 \quad x \neq-1 \quad x \neq 1
$$

Begin by doing a Sign Chart Analysis on $\frac{x}{x^{2}-1}$.
The first step is to factor completely.

$$
\frac{x}{x^{2}-1}=\frac{x}{(x-1)(x+1)}
$$

It is to this expression that we now apply the method.

Solutions to Exercises (continued)


Taking the blue line as our solution, and keeping in mind that $x \neq-1$ and $x \neq 1$ we get

$$
\begin{aligned}
\operatorname{Dom}(p) & =\left\{x \left\lvert\, \frac{x}{x^{2}-1} \geq 0\right., x \neq-1, x \neq 1\right\} & & \triangleleft \text { initial description } \\
& =\{x \mid-1<x \leq 0 \text { or } x>1\} & & \triangleleft \text { set notation } \\
& =(-1,0] \cup(1,+\infty) & & \triangleleft \text { interval notation }
\end{aligned}
$$

Solutions to Exercises (continued)
8.12. Solutions:
(a) $f(x)=4 x-1$

Set up: $\quad f(x)=0$

$$
4 x-1=0 \quad \triangleleft \text { substitute }
$$

Solve: $\quad 4 x=1 \quad \triangleleft$ add 1 to both sides $x=\frac{1}{4} \quad \triangleleft$ divide by 4

Presentation of Answer: The $x$-intercept is $x=\frac{1}{4}$
(b) $f(x)=x^{2}-3 x+2$

Set up: $\quad f(x)=0$

$$
x^{2}-3 x+2=0 \quad \triangleleft \text { substitute }
$$

Solve: $(x-2)(x-1)=1 \quad \triangleleft$ factor $x=1,2 \quad \triangleleft$ solved!

Presentation of Answer: The $x$-intercepts are $x=1,2$

Solutions to Exercises (continued)
(c) $f(x)=x^{3}+6 x^{2}++8 x$

Set up: $\quad f(x)=0$

$$
x^{3}+6 x^{2}+8 x=0 \quad \triangleleft \text { substitute }
$$

Solve: $x(x+2)(x+4)=1 \quad \triangleleft$ factor

$$
x=0,-2,-4 \quad \triangleleft \text { solved! }
$$

Presentation of Answer: The $x$-intercepts are $x=0,-2,-4$
(d) $f(x)=x^{2}+x+1$

Set up:

$$
f(x)=0
$$

$$
x^{2}+x+1=0 \quad \triangleleft \text { substitute (quadratic equation) }
$$

Solve: $\quad 1^{2}-4(1)(1)<0 \quad \triangleleft$ negative discriminant
No Solutions
Presentation of Answer: The function does not cross the $x$-axis.

Solutions to Exercises (continued)
(e) $f(x)=2 x^{2}-x-1$

Set up: $\quad f(x)=0$

$$
2 x^{2}-x-1=0 \quad \triangleleft \text { substitute }
$$

Solve: $(-1)^{2}-4(2)(-1)=9>0 \quad \triangleleft$ positive discriminant

$$
\begin{aligned}
x & =\frac{1 \pm \sqrt{9}}{4} & \triangleleft \text { quadratic formula } \\
& =\frac{1 \pm 3}{4} & \triangleleft \text { simplify } \\
& =-\frac{1}{2}, 1 &
\end{aligned}
$$

Note: This equation could have also been solved by factoring the left-hand side.
Presentation of Answer: The $x$-intercepts are $x=-\frac{1}{2}, 1$

Solutions to Exercises (continued)
(f) $f(x)=x^{2}+x-3$

Set up: $\quad f(x)=0$

$$
x^{2}+x-3=0 \quad \triangleleft \text { substitute }
$$

Solve: $1^{2}-4(1)(-3)=13>0$

$$
x=\frac{-1 \pm \sqrt{13}}{2} \triangleleft \text { quadratic formula }
$$

Presentation of Answer: The $x$-intercepts are $x=\frac{-1 \pm \sqrt{13}}{2}$
Exercise 8.12.
8.13. Answer: The $x$-intercept of a function $y=f(x)$ is the intersection the graph of $f$ and the $x$-axis. The $x$-axis is the graph of the function $g(x)=0$. Thus the $x$-axis is the intersection of two curves: $y=f(x)$ and $g(x)=0$.
8.14. Solutions: Hopefully, you used standard procedures.
(a) Find points of intersection: $f(x)=6 x+3$ and $g(x)=2 x-7$.

$$
\text { Set up: } \begin{array}{rlrl}
f(x) & =g(x) & & \triangleleft \text { equate ordinates } \\
\text { Solve: } \quad \begin{aligned}
& x+3 & =2 x-7 & \\
& =-10 & & \triangleleft \text { substitute } \\
& & & \triangleleft \text { add }-2 x-3 \text { both sides } \\
x & =-\frac{10}{4} & & \triangleleft \text { divide by } 4 \\
& =-\frac{5}{2} & & \triangleleft \text { done }!
\end{aligned} r l
\end{array}
$$

When $x=-\frac{5}{2}, f\left(-\frac{5}{2}\right)=6\left(-\frac{5}{2}\right)+3=-12$.
Presentation of Answer: $P\left(-\frac{5}{2},-12\right)$
(b) Find points of intersection: $f(x)=x+3$ and $g(x)=2-8 x$.

Set up: $f(x)=g(x) \quad \triangleleft$ equate ordinates

$$
x+3=2-8 x \quad \triangleleft \text { substitute }
$$

Solve: $\quad 9 x=-1 \quad \triangleleft$ add $8 x-3$ both sides

$$
x=-\frac{1}{9} \quad \triangleleft \text { divide by } 9
$$

Solutions to Exercises (continued)
When $x=-\frac{1}{9}, f\left(-\frac{1}{9}\right)=-\frac{1}{9}+3=\frac{26}{9}$.
Presentation of Answer: $P\left(-\frac{1}{9}, \frac{26}{9}\right)$
(c) Find points of intersection: $f(x)=x^{2}+7 x-1$ and $g(x)=4 x-3$.

Set up:

$$
f(x)=g(x) \quad \triangleleft \text { equate ordinates }
$$

$$
x^{2}+7 x-1=4 x-3 \quad \triangleleft \text { substitute }
$$

Solve:

$$
\begin{aligned}
x^{2}+3 x+2 & =0 & & \triangleleft \text { add }-4 \\
(x+1)(x+2) & =0 & & \triangleleft \text { factor } \\
x & =-1,-2 & & \triangleleft \text { done }
\end{aligned}
$$

Calculation of Ordinates: When $x=-1, y=g(2)=-7$. When $x=-2, y=g(3)=-11$.
Presentation of Answer:
Points of Intersection: $P_{1}(-1,-7), P_{2}(-2,-11)$
Exercise 8.14.

Solutions to Exercises (continued)
8.15. Solutions: We use standard procedures around here ... how about you?
(a) Find points of intersection: $f(x)=2 x^{2}-5 x+2$ and $g(x)=x+3$.
Set up: $\quad f(x)=g(x) \quad \triangleleft$ equate ordinates

$$
2 x^{2}-5 x+2=x+3 \quad \triangleleft \text { substitute }
$$

Solve: $\quad 2 x^{2}-6 x-1=0 \quad \triangleleft$ add $-x-3$ both sides

$$
\begin{aligned}
(-6)^{2}-4(2)(-1) & =44>0 \\
x & =\frac{6 \pm \sqrt{44}}{4} \\
& \triangleleft \text { pos. discrim. } \\
& =\frac{3 \pm \sqrt{11}}{2}
\end{aligned}
$$

These two curves intersect at $x=\frac{3-\sqrt{11}}{2}, \frac{3+\sqrt{11}}{2}$

Solutions to Exercises (continued)
(b) Find points of intersection: $f(x)=x^{2}+4 x-1$ and $g(x)=$ $1-4 x-x^{2}$.

Set up:

$$
f(x)=g(x) \quad \triangleleft \text { equate ordinates }
$$

$$
x^{2}+4 x-1=1-4 x-x^{2} \quad \triangleleft \text { substitute }
$$

Solve: $2 x^{2}+8 x-2=0$
$\triangleleft$ add $-1+4 x+x^{2}$

$$
8^{2}-4(2)(-2)=80>0
$$

$$
\triangleleft \text { pos. discrim. }
$$

$$
\begin{aligned}
x & =\frac{-8 \pm \sqrt{80}}{4} \quad \triangleleft \text { Quadratic formula } \\
& =\frac{-8 \pm 4 \sqrt{5}}{4} \\
& =-2 \pm \sqrt{5}
\end{aligned}
$$

These two curves intersect at $x=-2-\sqrt{5},-2+\sqrt{5}$

Solutions to Exercises (continued)
(c) Find points of intersection: $f(x)=3 x^{2}+1$ and $g(x)=x^{2}-5 x$.

$$
\text { Set up: } \begin{aligned}
f(x) & =g(x) & & \triangleleft \text { equate ordinates } \\
3 x^{2}+1 & =x^{2}-5 x & & \triangleleft \text { substitute } \\
\text { Solve: } 2 x^{2}+5 x+1 & =0 & & \triangleleft \text { add }-1+4 x+x^{2} \\
5^{2}-4(2)(1) & =17>0 & & \triangleleft \text { pos. discrim. } \\
x & =\frac{-5 \pm \sqrt{17}}{4} & & \triangleleft \text { Quadratic formula }
\end{aligned}
$$

These two curves intersect at $x=\frac{-5-\sqrt{17}}{4}, \frac{-5+\sqrt{17}}{4}$
Exercise 8.15.

Solutions to Exercises (continued)
8.16. Solutions: Follow the procedure.
(a) $f(x)=4 x-2$ and $g(x)=4 x+12$

Set up: $\quad f(x)=g(x) \quad \triangleleft$ equate ordinates $4 x-2=4 x+12 \quad \triangleleft$ substitute Solve: $\quad 0=14 \quad \triangleleft$ add $-4 x+2$ both sides

The equation $0=14$ has no solution; i.e., no value of $x$ can satisfy the equation $0=14$. Therefore, these two curves do not intersect.
(b) $f(x)=2 x^{2}+3$ and $g(x)=x^{2}-1$.

Set up: $\quad f(x)=g(x) \quad \triangleleft$ equate ordinates

$$
2 x^{2}+3=x^{2}-1 \quad \triangleleft \text { substitute }
$$

Solve: $\quad x^{2}=-4 \quad \triangleleft$ add $-x^{2}+1$ both sides
The equation $x^{2}=-4$ has no solutions; therefore, these two curves do not intersect.

Solutions to Exercises (continued)
(c) $f(x)=x^{2}+2 x+2$ and $g(x)=x+1$

Set up: $\quad f(x)=g(x) \quad \triangleleft$ equate ordinates

$$
x^{2}+2 x+2=x+1 \quad \triangleleft \text { substitute }
$$

Solve: $\quad x^{2}+x+1=0 \quad \triangleleft$ add $-x-3$ both sides

$$
(1)^{2}-4(1)(1)=-3<0 \quad \triangleleft \text { negative discriminant }
$$

A negative discriminant $\left(b^{2}-4 a c<0\right)$ implies that the equation has not solution; therefore, these two equations do not intersect.
(d) $f(x)=x^{2}-2 x-4$ and $g(x)=4 x^{2}-3$

Set up: $\quad f(x)=g(x) \quad \triangleleft$ equate ordinates

$$
x^{2}-2 x-4=4 x^{2}-3 \quad \triangleleft \text { substitute }
$$

Solve: $\quad-3 x^{2}-2 x-1=0 \quad \triangleleft$ add $-4 x^{2}+3$

$$
(-2)^{2}-4(-3)(-1)=-8<0 \quad \triangleleft \text { negative discriminant }
$$

A negative discriminant implies the equation has no solution.

## Solutions to Examples

8.1. Solution:

$$
\begin{aligned}
& P:\left(x_{1}, y_{1}\right)=(-2,4) \\
& Q:\left(x_{2}, y_{2}\right)=(3,-1)
\end{aligned}
$$

We take the difference in the first coordinates and the difference in the second coordinates.

$$
\begin{aligned}
x_{1}-x_{2} & =-2-3=-5 \\
y_{1}-y_{2} & =4-(-1)=4+1=5
\end{aligned}
$$

We now take the sum of the squares of these two:

$$
\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}=(-5)^{2}+5^{2}=25+25=50 .
$$

Finally, we take the square root of this result:

$$
d(P, Q)=\sqrt{\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}}=\sqrt{50}=5 \sqrt{2}
$$

Presentation of Solution: $d(P, Q)=5 \sqrt{2}$

Solutions to Examples (continued)
Of course, this process can be accelerated once you fully understand the computational steps.
8.2. Solution to: (a) Define $f(x)=x^{2}+3 x+1$. The natural domain is the set of all real numbers for which the value of $f(x)=x^{2}+3 x+1$ can be computed as a real number. For any real number $x$, the expression $x^{2}+3 x+1$ evaluates to a real number. Therefore, we deduce,

$$
\operatorname{Dom}(f)=\mathbb{R}=(-\infty,+\infty)
$$

Solution to: (b) Define $g(x)=\frac{x^{2}}{x^{2}-3 x+2}$. The numerator and denominator always evaluate to a real number; however, if the denominator evaluates to zero, the quotient is not a real number. Thus, we can say that

$$
\operatorname{Dom}(g)=\left\{x \mid x^{2}-3 x+2 \neq 0\right\}
$$

This should not be considered to be a satisfactory characterization of the domain of $g$ though.

Solutions to Examples (continued)
First find where $x^{2}-3 x+2=0$, and reason from there. Solve

$$
\begin{aligned}
x^{2}-3 x+2 & =0 & & \triangleleft \text { given } \\
(x-1)(x-2) & =0 & & \triangleleft \text { factor } \\
x & =1,2 & & \triangleleft \text { solve }
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\operatorname{Dom}(g) & =\left\{x \mid x^{2}-3 x+2 \neq 0\right\} & & \\
& =\{x \mid x \neq 1, x \neq 2,\} & & \triangleleft \text { set notation } \\
& =(-\infty, 1) \cup(1,2) \cup(2,+\infty) & & \triangleleft \text { interval notation }
\end{aligned}
$$

Solution to: (c) Define $h(x)=\sqrt{x+2}$. For any $x, x+2$ evaluates to a real number, but for $\sqrt{x+2}$ to evaluate to a real number we must have $x+2 \geq 0$. Thus,

$$
\operatorname{Dom}(h)=\{x \mid x+2 \geq 0\}
$$

Solutions to Examples (continued)

Again, we should not be satisfied with this formulation. We next solve the inequality:

$$
x+2 \geq 0 \Longrightarrow x \geq-2
$$

Thus,

$$
\operatorname{Dom}(h)=\{x \mid x \geq-2\}=[-2,+\infty)
$$

Solution to: (d) Define $p(x)=\sqrt{x^{2}-1}$. In order for $p(x)$ to evaluate to a real number, we require $x^{2}-1 \geq 0$ and $x \neq 0$. Thus,

$$
\operatorname{Dom}(p)=\left\{x \mid x^{2}-1 \geq 0\right\}
$$

We need to solve the inequality $x^{2}-1 \geq 0$. To do this, we use the Sign Chart Method originally discussed in Lesson 7. (Actually, this method really isn't needed for this simple inequality. We could solve as follows:

$$
x^{2}-1 \geq 0 \Longrightarrow x^{2} \geq 1 \Longrightarrow|x| \geq 1
$$

but we shall use the Sign Chatr Method in any case, just to remind you of this method.)

Solutions to Examples (continued)

We begin by factoring completely the left-hand side (which is a difference of squares):

$$
(x+1)(x-1) \geq 0
$$

The Sign Chart of $(x+1)(x-1)$


Therefore, the solution to the inequality $x^{2}-1 \geq 0$ is

$$
(-\infty,-1] \cup[1,+\infty)
$$

But the solution to this inequality is the natural domain of $p$. Thus,

$$
\operatorname{Dom}(p)=(-\infty,-1] \cup[1,+\infty)
$$

Notice how all the techniques of algebra (LESSONS 1-7) are used: factoring, solving inequalities, interval notation and so on.

This is the discouraging and challanging thing about mathematics: To solve any given problem, we must call on our entire history of experiences in mathematics. This is why it is so important for us to try to master each of the little steps we take toward our final goals.

Example 8.2.

Solutions to Examples (continued)
8.3. Solution: Consider $f(x)=\sqrt{\frac{x}{x+1}}$. Based on the above strategy, we see that

$$
\operatorname{Dom}(f)=\left\{x \mid x \neq-1 \text { and } \frac{x}{x+1} \geq 0\right\}
$$

The first condition, $x \neq-1$, avoids having zero in the denominator (we exclude $x=-1$ from the domain); the second condition is necessary for the radicand to be nonnegative. (The square root of a nonnegative number is a real number, whereas the square root of a negative number is a complex number. We don't want to work with complex numbers at this time.)

As you can see, I've simply translated the strategy into a series of inequalities. We solve the inequality

$$
\frac{x}{x+1} \geq 0
$$

first using the Sign Chart Methods.

Solutions to Examples (continued)


Therefore,

$$
\frac{x}{x+1} \geq 0 \Longrightarrow x \leq-1 \text { or } x \geq 0
$$

But, we also have the condition $x \neq-1$; this changes the above solution slightly to

$$
x<-1 \text { or } x \geq 0 .
$$

Presentation of Answer:

$$
\operatorname{Dom}(f)=(-\infty,-1) \cup[0,+\infty)
$$

## Solutions to Examples (continued)

Example 8.3.

Solutions to Examples (continued)
8.4. Solutions: We follow the standard procedures.
(a) Find the points of intersection of $f(x)=3 x+2$ and $g(x)=5 x-4$.

$$
\begin{array}{rlrl}
\text { Set up: } & f(x) & =g(x) & \\
& & \triangleleft \text { equate ordinates } \\
\text { Solve: } \quad-2 x & =5 x-4 & & \triangleleft \text { substitute } \\
& & & \triangleleft \text { add }-5 x-2 \text { to both sides } \\
x & =3 & & \triangleleft \text { divide by } 3
\end{array}
$$

At $x=3, f(3)=3(3)+2=11$. Thus, $(3,11)$ is the point of intersection.
Presentation of Solution: Intersection Point(s): $P(3,11)$

Solutions to Examples (continued)
(b) Find intersection points of $f(x)=x^{2}-3 x+1$ and $g(x)=2 x-5$.

$$
\text { Set up: } \quad f(x)=g(x) \quad \triangleleft \text { equate ordinates }
$$

$$
x^{2}-3 x+1=2 x-5 \quad \triangleleft \text { substitute }
$$

Solve: $\quad x^{2}-5 x+6=0 \quad \triangleleft$ add $-2 x+5$ both sides

$$
\begin{aligned}
(x-2)(x-3) & =0 & & \triangleleft \text { factor } \\
x & =2,3 & & \triangleleft \text { done }
\end{aligned}
$$

Calculation of Ordinates: When $x=2, y=g(2)=-1$. When $x=3, y=g(3)=1$.
Presentation of Answer:

$$
\text { Points of Intersection: } P_{1}(2,-1), P_{2}(3,1)
$$

## Important Points

This terminology enables us to refer to the horizontal and vertical axes (and the first and second coordinates) in a manner that is independent of the labeling of the axis. Sometimes the axes are labeled by using other letters such as $s, t, u$ or $v$; in these cases, when we discuss the $x$-axis, for example, we may not have an $x$-axis.

Of course, the letters $x$ and $y$ are symbols representing the horizontal and vertical axes. Whatever we say about the " $x$-axis" we are saying about the horizontal axis, whatever its name.

Sometimes it is convenient to refer to the axes without referring to a specific coordinate axis label.

The correct answer is (b): $Q$ is between $P$ and $R$ if and only if the distance from $P$ to $R$ equals the combined distances from $P$ to $Q$ and $Q$ to $R$. This statement is summarized in the equation

$$
d(P, R)=d(P, Q)+d(Q, R)
$$

Comments: Statements (a) and (c) imply the three points form a right triangle ... that's not what we want. Statement (d) means $R$ is between $P$ and $Q \ldots$ close but not what we wanted either.

To understand the solution, draw a picture of three colinear points in the plane. Label the two extreme points $P$ and $R$ and the one between them $Q$. Observe that $d(P, R)=d(P, Q)+d(Q, R)$.

To understand my comments, draw pictures in the plane to reflect each alternative.

Solutions to the Quiz. The basic tool is the horizontal distance formula (2).

1. Yes. Since the ordinates of the two points $(2,6)$ and $(-4,6)$ are equal, this means, by (2), they are horizontally oriented.
2. Answer (c). The distance between the points $P(3,-9)$ and $Q(8,-9)$ is

$$
d(P, Q)=\left|x_{1}-x_{2}\right|=|3-8|=|-5|=5
$$

3. Answer (d). The distance between the points $P(6,-3)$ and $Q(-3,-3)$ is

$$
d(P, Q)=\left|x_{1}-x_{2}\right|=|6-(-3)|=|6+3|=9
$$

Notice the use of the parentheses to properly evaluate the formula.
4. Answer (a). The distance between the points $P(-5,2)$ and $Q(-9,2)$ is

$$
d(Q, P)=\left|x_{1}-x_{2}\right|=|-5-(-9)|=|-5+9|=|9-5|=4
$$

Important Points (continued)
Again, note the use of parentheses to properly evaluate the formula. Of course, $d(P, Q)=d(Q, P)-\mathrm{I}$ hope that didn't bother you.

Did you get $100 \%$ ? I hope so.

Let the Cartesian coordinates of $M$ be denoted by $M(\bar{x}, \bar{y})$. We can compute the values of $\bar{x}$ and $\bar{y}$ by simply describing the geometric properties of $M$ :

$$
\begin{aligned}
d(P, Q) & =d(P, M)+d(M, Q) & & \triangleleft M \text { is between } P \text { and } Q \\
d(P, M) & =\frac{1}{2} d(P, Q) & & \triangleleft M \text { is halfway between }
\end{aligned}
$$

We have two equations and two unknowns ( $\bar{x}$ and $\bar{y}$ ), we can solve for $\bar{x}$ and $\bar{y}$. However, this method is rather messy.

The enthusiastic student can pursue this train of thought.

## Models for Functions.

In this section we present different ways of thinking about functions that may be of help to you.

## $\triangleright$ A Function as a Mapping.

One traditional way of looking at a function is as a mapping or a transformation. Let $f: A \rightarrow B$ be a function, and let $x \in A$. As discussed above, $y=f(x)$ is the value of the function at $x$. We can also look upon $f$ as a mapping or transformation: $f$ maps $x$ onto $y$, or, $y$ is the image of $x$ under $f$.

This interpretation is one of the origins of the notation introduced above:

$$
x \stackrel{f}{\stackrel{f}{b}} y .
$$

Try to get the feeling for this interpretation. Imagine a bunch of arrows pointing from elements $x$ in the set $A$ to elements $y$ in the set $B$. The arrows point from each $x$ to the corresponding value of $y$, as the "arrow notation" above suggests. When we see $x$ we immediately think of its
corresponding value $f(x)$. The Venn Diagram, described next, is a more visual representation.
$\triangleright$ Venn Diagram of a Function.

$\downarrow$Figure ping, transformation) is given. This graph represents $f$ as it $A$ into the co-domain set $B$. The image of $x$ under this map, $f$, is denoted by $y$ in the figure. Visualize a function as a bunch of "arrows" pointing from set $A$ into set $B$. The tail of a typical arrow is at $x$, and the arrow "points" to the corresponding $y$-value.

This model is very useful in understanding functions and various operations performed on functions (such as composition of functions).

$\rightleftharpoons$To further illustrate the point, Figure I-2 depicts a relation that is not a function. A function is a rule that associates Figure I-2 with each value $x$ is a certain domain set, a corresponding unique $y$-value. A rule that associates with at least one $x$ more than one corresponding $y$-value would not be a function-as illustrated
in Figure I-2. Observe that associated with $x$ is two corresponding values-labeled $y$ and $z$.
As a particular example of this, consider the equation: $x^{2}+y^{2}=1$. For $x=0$, there are two values of $y$ that satisfy this equation: $y=1$ and $y=-1$. This equation does not define, therefore, $y$ as a function of $x$. (Visualize two arrow coming out of $x=0$, one pointing to $y=1$ and the other pointing to $y=-1$.

## $\triangleright$ A Function as a Black Box.

This interpretation of function is often associated with the engineering world. A function is like a machine (a black box). We have a machine (a black box) that takes input into it, and, as a result, yields output. The black box is the function, the input are the values in the domain of the function, and the output of the box (function) are the values in the range of the function.

$$
x \longrightarrow \text { function } \longrightarrow y
$$

Actually, this looks more like a white box to me :=\{).

A black box you are familiar with is the hand-held calculator. This is usually, literally, a black box. You input $x$-values on the key pad, say $x=12$. You then choose the black box to input this value of $x$ into. Your calculator is actually made up of a large number of black boxes - called function keys (Hey, function!). Choose the function key labeled

$$
x^{2}
$$

and press it - out comes the output. You will see (on your real or imagined display panel) the value 144 .

This is a representation of the black box model.

$$
x \longrightarrow x^{2} \longrightarrow x^{2}
$$

put $x=12$,

$$
12 \longrightarrow x^{2} \longrightarrow 144 .
$$

Input-output, input-output - and that's the way it works.


[^0]:    ${ }^{1}$ Here is a brief discussion of an alternate approach.

